Chapter 8. Radiation from Moving Charges

Notes:

- *Most of the material presented in this chapter is taken from Jackson, Chap. 12 and 14, and Rybicki and Lightman, Chap. 4.*
- *In this chapter, we will be using Gaussian units for the Maxwell equations and other related mathematical expressions.*
- *Latin indices are used for space coordinates only (e.g., $i = 1,2,3$, etc.), while Greek indices are for space-time coordinates (e.g., $\alpha = 0,1,2,3$, etc.).*

8.1 Solution of the Wave Equation in Covariant Form

In this chapter, we will solve for the electromagnetic fields, in a covariant manner, through the use of the four-potential. We will do so in a way similar to what was done in Chapter 4 in analyzing the solution of the wave equation for the potentials. The differences between to the two analyses reside in the facts that in this chapter we will use a covariant treatment, and solve the wave equation through the (invariant) Green function method (a generalization of what was done in Section 4.3.2).

Our starting point is equation (7.96) for the wave equation of the four-potential A^β in the Lorenz gauge (i.e., $\partial_\beta A^\beta = 0$). More precisely, we wish to solve

$$\Box A^\beta = \frac{4\pi}{c} J^\beta, \tag{8.1}$$

for a general four-current distribution $J^\beta(\vec{\mathbf{x}})$. We can accomplish this by first solving for the Green function $D(\vec{\mathbf{x}},\vec{\mathbf{x}}')$ with

$$\Box_x D(\vec{\mathbf{x}},\vec{\mathbf{x}}') = \delta(\vec{\mathbf{x}}-\vec{\mathbf{x}}'), \tag{8.2}$$

and then find the components of the four-potential using

$$A^\beta(\vec{\mathbf{x}}) = \frac{4\pi}{c}\int D(\vec{\mathbf{x}},\vec{\mathbf{x}}') J^\beta(\vec{\mathbf{x}}')\, d^4x'. \tag{8.3}$$

Since the problem is “spherically” symmetric (that is, in four dimensions), then the Green function can only depend on the difference $\vec{\mathbf{z}} \equiv \vec{\mathbf{x}}-\vec{\mathbf{x}}'$, and we write therefore $D(\vec{\mathbf{x}},\vec{\mathbf{x}}') = D(\vec{\mathbf{x}}-\vec{\mathbf{x}}') = D(\vec{\mathbf{z}})$. We now operate on both sides of equation (8.2) with a Laplace transform (see the Appendix on the subject) for the time component, and a three-dimensional Fourier transform for the three-space components. Doing so, we find

$$\left(s^2 + k^2\right) D(s,\mathbf{k}) = 1, \tag{8.4}$$

since $\Box = \partial_0 \partial^0 - \nabla^2$. Please note that, in this case, the Laplace transform links the component z_0 (not the time) to the s-domain . Alternatively, we can write

$$D(s,\mathbf{k}) = \frac{1}{\left(s^2 + k^2\right)} = \frac{1}{(s - ik)(s + ik)}, \tag{8.5}$$

and we are ready to apply the inverse transforms to recover $D(\vec{\mathbf{z}})$. By first computing the inverse Laplace transform (with the residue theorem), we find

$$\begin{aligned} D(\vec{\mathbf{z}}) &= \frac{H(z_0)}{(2\pi)^3} \int d^3k \; e^{i\mathbf{k}\cdot\mathbf{z}} \left(\frac{e^{ik_0 z_0}}{2ik} - \frac{e^{-ik_0 z_0}}{2ik} \right) \\ &= \frac{H(z_0)}{(2\pi)^3} \int d^3k \; \frac{e^{i\mathbf{k}\cdot\mathbf{z}}}{k} \sin(k_0 z_0), \end{aligned} \tag{8.6}$$

with $H(z_0)$ the Heaviside step function (see equation (1.14)). We now change to spherical coordinate (in three-space, this time) where $d^3k = dk\, d\theta\, d\varphi \; k^2 \sin(\theta)$, then

$$\begin{aligned} D(\vec{\mathbf{z}}) &= \frac{H(z_0)}{(2\pi)^2} \int_0^\infty dk \; k \sin(k_0 z_0) \int_{-1}^{1} d\left[\cos(\theta)\right] e^{ikR\cos(\theta)} \\ &= \frac{H(z_0)}{2\pi^2 R} \int_0^\infty dk \; \sin(k_0 z_0) \sin(kR), \end{aligned} \tag{8.7}$$

with $R = |\mathbf{z}| = |\mathbf{x} - \mathbf{x}'|$. Expanding the sine functions with complex exponentials we have

$$D(\vec{\mathbf{z}}) = \frac{H(z_0)}{8\pi^2 R} \int_0^\infty dk \left(e^{ik(R - z_0)} + e^{-ik(R - z_0)} - e^{ik(R + z_0)} - e^{-ik(R + z_0)} \right), \tag{8.8}$$

and making the change of variable $k \to -k$ for the integrals where the imaginary argument of the corresponding exponential is negative, we then find

$$\begin{aligned} D(\vec{\mathbf{x}} - \vec{\mathbf{x}}') &= \frac{H(x_0 - x_0')}{8\pi^2 R} \int_{-\infty}^{\infty} dk \left(e^{ik\left[R - (x_0 - x_0')\right]} - e^{ik\left[R + (x_0 - x_0')\right]} \right) \\ &= \frac{H(x_0 - x_0')}{4\pi R} \left\{ \delta\left[R - (x_0 - x_0')\right] - \delta\left[R + (x_0 - x_0')\right] \right\}. \end{aligned} \tag{8.9}$$

But since $R > 0$ and $x_0 - x_0' > 0$, then the Green function is

$$D(\vec{\mathbf{x}} - \vec{\mathbf{x}}') = \frac{H(x_0 - x_0')}{4\pi R} \delta\left[R - (x_0 - x_0')\right]. \tag{8.10}$$

Alternatively, equation (8.10) can be transformed with

$$\begin{aligned}\delta\left[\left(\vec{\mathbf{x}}-\vec{\mathbf{x}}'\right)^2\right] &= \delta\left[\left(x_0-x_0'\right)^2-\left|\mathbf{x}-\mathbf{x}'\right|^2\right] \\ &= \delta\left[\left(x_0-x_0'-R\right)\left(x_0-x_0'+R\right)\right] \\ &= \frac{1}{2R}\left[\delta\left(x_0-x_0'-R\right)+\delta\left(x_0-x_0'+R\right)\right],\end{aligned} \tag{8.11}$$

where we used equation (1.13)

$$\delta\left[f(x)\right]=\sum_i \frac{\delta\left(x-x_i\right)}{\left|\left(\frac{df}{dx}\right)_{x=x_i}\right|} \tag{8.12}$$

and

$$\left.\frac{d\left[\left(\vec{\mathbf{x}}-\vec{\mathbf{x}}'\right)^2\right]}{d\left|x_0-x_0'\right|}\right|_{x_0-x_0'=\pm R} = 2\left|x_0-x_0'\right|\Big|_{x_0-x_0'=\pm R}=2R. \tag{8.13}$$

Furthermore, since only one of the two delta functions (i.e., the first) in equation (8.11), for reasons discussed earlier, contributes to the Green function, then

$$\boxed{D\left(\vec{\mathbf{x}}-\vec{\mathbf{x}}'\right)=\frac{1}{2\pi}H\left(x_0-x_0'\right)\delta\left[\left(\vec{\mathbf{x}}-\vec{\mathbf{x}}'\right)^2\right]} \tag{8.14}$$

8.2 The Liénard-Wiechert Potentials and Fields for a Point Charge

We now consider the potentials and fields due to a single moving particle of charge q. If the position of the particle in a given inertial frame K is $\mathbf{r}(t)$, then its charge and current densities in that frame are

$$\begin{aligned}\rho(\mathbf{x},t) &= q\delta\left[\mathbf{x}-\mathbf{r}(t)\right] \\ \mathbf{J}(\mathbf{x},t) &= q\mathbf{u}(t)\delta\left[\mathbf{x}-\mathbf{r}(t)\right],\end{aligned} \tag{8.15}$$

where $\mathbf{u}(t)=d\mathbf{r}(t)/dt$ is the velocity of the charge in K. Using the definition for the four-current (see equation (7.91)), equations (8.15) can be combined into a single covariant relation with

$$J^{\alpha} = qc\int U^{\alpha}(\tau)\delta\left[\vec{\mathbf{x}} - \vec{\mathbf{r}}(\tau)\right] d\tau, \tag{8.16}$$

where U^{α} is the charge's four-velocity, and r^{α} its four-position (c.f., equation (6.13) of the second problem list). In K, we have

$$U^{\alpha} = \begin{pmatrix} \gamma c \\ \gamma \mathbf{u} \end{pmatrix}, \quad \text{and} \quad r^{\alpha}(\tau) = \begin{pmatrix} ct \\ \mathbf{r}(t) \end{pmatrix}. \tag{8.17}$$

Substituting equations (8.16) and (8.14) into equation (8.3), we have for the four-potential

$$\begin{aligned} A^{\alpha} &= 2q\int d\tau\, U^{\alpha}(\tau)\int d^4x'\, H\left(x_0 - x_0'\right)\delta\left[\left(\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right)^2\right]\delta\left[\vec{\mathbf{x}}' - \vec{\mathbf{r}}(\tau)\right] \\ &= 2q\int d\tau\, U^{\alpha}(\tau)H\left[x_0 - r_0(\tau)\right]\delta\left\{\left[\vec{\mathbf{x}} - \vec{\mathbf{r}}(\tau)\right]^2\right\}, \end{aligned} \tag{8.18}$$

where the integral gives a contribution from the "retarded" proper time $\tau = \tau_0$ (since from the argument of the Heaviside function $x_0 > r_0(\tau)$) defined by

$$\left[\vec{\mathbf{x}} - \vec{\mathbf{r}}(\tau)\right]^2 = 0. \tag{8.19}$$

We use once more equation (1.13) (or (8.12)) with

$$\frac{d}{d\tau}\left\{\left[\vec{\mathbf{x}} - \vec{\mathbf{r}}(\tau)\right]^2\right\} = -2\left[\vec{\mathbf{x}} - \vec{\mathbf{r}}(\tau)\right]_{\gamma} U^{\gamma}(\tau), \tag{8.20}$$

and

$$\delta\left\{\left[\vec{\mathbf{x}} - \vec{\mathbf{r}}(\tau)\right]^2\right\} = \frac{\delta(\tau - \tau_0) + \delta(\tau + \tau_0)}{2\left[\vec{\mathbf{x}} - \vec{\mathbf{r}}(\tau)\right]_{\gamma} U^{\gamma}(\tau)}, \tag{8.21}$$

where it is understood that these relations are evaluated at τ_0. Since only the first of the two delta functions in equation (8.21) contributes to the integral of equation (8.18), the four-potential becomes

$$\boxed{A^{\beta}(\vec{\mathbf{x}}) = \left.\frac{qU^{\beta}(\tau)}{\vec{\mathbf{U}} \cdot \left[\vec{\mathbf{x}} - \vec{\mathbf{r}}(\tau)\right]}\right|_{\tau=\tau_0}} \tag{8.22}$$

These potentials are known as the **Liénard-Wiechert** potentials. In the inertial frame K, we can expand the denominator using equations (8.17) to get

$$\vec{\mathbf{U}}\cdot\left[\vec{\mathbf{x}}-\vec{\mathbf{r}}(\tau_0)\right]=U_0\left[x_0-r_0(\tau_0)\right]-\mathbf{U}\cdot\left[\mathbf{x}-\mathbf{r}(\tau_0)\right]$$
$$=\gamma cR-\gamma\mathbf{u}\cdot\mathbf{n}R \tag{8.23}$$
$$=\gamma cR(1-\boldsymbol{\beta}\cdot\mathbf{n}),$$

where $R\equiv x_0-r_0(\tau_0)=\left|\mathbf{x}-\mathbf{r}(\tau_0)\right|$, and $\mathbf{n}$ is the unit vector in the direction of $\mathbf{x}-\mathbf{r}(\tau_0)$, and $\boldsymbol{\beta}=\mathbf{u}(\tau)/c$. We are now in a position to give expressions for the scalar and vector potentials in K

$$\boxed{\begin{aligned}\Phi(\mathbf{x},t)&=\left[\frac{q}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{\text{ret}}\\ \mathbf{A}(\mathbf{x},t)&=\left[\frac{q\boldsymbol{\beta}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{\text{ret}}\end{aligned}} \tag{8.24}$$

where "ret" means that the potentials are to be evaluated at the retarded proper time τ_0, given by $r_0(\tau_0)=x_0-R$. The electromagnetic fields could be evaluated from equations (8.24) (or equations (8.22)), but we will use instead equation (8.18) as a starting point, and apply equation (7.96). Thus, we need to evaluate the following

$$\begin{aligned}\partial^\alpha A^\beta&=2q\int d\tau\, U^\beta(\tau)\Big(\partial^\alpha H\left[x_0-r_0(\tau)\right]\delta\left\{\left[\vec{\mathbf{x}}-\vec{\mathbf{r}}(\tau)\right]^2\right\}\\&\quad+H\left[x_0-r_0(\tau)\right]\partial^\alpha\delta\left\{\left[\vec{\mathbf{x}}-\vec{\mathbf{r}}(\tau)\right]^2\right\}\Big)\\&=2q\int d\tau\, U^\beta(\tau)\Big(\delta\left[x_0-r_0(\tau)\right]\delta\left\{\left[\vec{\mathbf{x}}-\vec{\mathbf{r}}(\tau)\right]^2\right\}\\&\quad+H\left[x_0-r_0(\tau)\right]\partial^\alpha\delta\left\{\left[\vec{\mathbf{x}}-\vec{\mathbf{r}}(\tau)\right]^2\right\}\Big)\\&=2q\int d\tau\, U^\beta(\tau)\Big(\delta\left[x_0-r_0(\tau)\right]\delta\left(-R^2\right)\\&\quad+H\left[x_0-r_0(\tau)\right]\partial^\alpha\delta\left\{\left[\vec{\mathbf{x}}-\vec{\mathbf{r}}(\tau)\right]^2\right\}\Big).\end{aligned} \tag{8.25}$$

We will not, however, consider in our analysis cases where $R=0$ (i.e., the observer is not located at the "retarded" position of the source), equation (8.25) then simplifies to

$$\partial^\alpha A^\beta=2q\int d\tau\, U^\beta(\tau)H\left[x_0-r_0(\tau)\right]\partial^\alpha\delta\left\{\left[\vec{\mathbf{x}}-\vec{\mathbf{r}}(\tau)\right]^2\right\}. \tag{8.26}$$

We will need to use the following to solve this integral

$$\partial^{\alpha}\delta\left[f(\tau)\right]=\partial^{\alpha}\left[f(\tau)\right]\frac{d}{df}\delta\left[f(\tau)\right]$$
$$=\partial^{\alpha}\left[f(\tau)\right]\frac{d\tau}{df}\frac{d\left\{\delta\left[f(\tau)\right]\right\}}{d\tau}. \tag{8.27}$$

With $f(\tau)=\left[\vec{\mathbf{x}}-\vec{\mathbf{r}}(\tau)\right]^2$ we have

$$\partial^{\alpha}\left\{\left[\vec{\mathbf{x}}-\vec{\mathbf{r}}(\tau)\right]^2\right\}=\partial^{\alpha}\left[\left(x^{\gamma}-r^{\gamma}\right)\left(x_{\gamma}-r_{\gamma}\right)\right]$$
$$=\delta^{\alpha\gamma}\left(x_{\gamma}-r_{\gamma}\right)+\delta^{\alpha}{}_{\gamma}\left(x^{\gamma}-r^{\gamma}\right) \tag{8.28}$$
$$=2\left(x^{\alpha}-r^{\alpha}\right),$$

and using equation (8.20) we find

$$\partial^{\alpha}\delta\left\{\left[\vec{\mathbf{x}}-\vec{\mathbf{r}}(\tau)\right]^2\right\}=-\frac{\left(x^{\alpha}-r^{\alpha}\right)}{\left[\vec{\mathbf{x}}-\vec{\mathbf{r}}(\tau)\right]_{\gamma}U^{\gamma}(\tau)}\frac{d}{d\tau}\delta\left\{\left[\vec{\mathbf{x}}-\vec{\mathbf{r}}(\tau)\right]^2\right\}. \tag{8.29}$$

Inserting equation (8.29) into equation (8.26) we get

$$\partial^{\alpha}A^{\beta}=-2q\int d\tau\, U^{\beta}(\tau)H\left[x_0-r_0(\tau)\right]\frac{\left[x^{\alpha}-r^{\alpha}(\tau)\right]}{\left[\vec{\mathbf{x}}-\vec{\mathbf{r}}(\tau)\right]_{\gamma}U^{\gamma}(\tau)}\frac{d}{d\tau}\delta\left\{\left[\vec{\mathbf{x}}-\vec{\mathbf{r}}(\tau)\right]^2\right\}$$
$$=2q\int d\tau\,\frac{d}{d\tau}\left\{\frac{U^{\beta}(\tau)\left[x^{\alpha}-r^{\alpha}(\tau)\right]H\left[x_0-r_0(\tau)\right]}{\left[\vec{\mathbf{x}}-\vec{\mathbf{r}}(\tau)\right]_{\gamma}U^{\gamma}(\tau)}\right\}\delta\left\{\left[\vec{\mathbf{x}}-\vec{\mathbf{r}}(\tau)\right]^2\right\}, \tag{8.30}$$

after integrating by parts, and

$$\partial^{\alpha}A^{\beta}=2q\int d\tau\left\{-U_0\frac{\left(x^{\alpha}-r^{\alpha}\right)U^{\beta}\delta\left(x_0-r_0\right)}{\left(x_{\gamma}-r_{\gamma}\right)U^{\gamma}}+\frac{d}{d\tau}\left[\frac{\left(x^{\alpha}-r^{\alpha}\right)U^{\beta}}{\left(x_{\gamma}-r_{\gamma}\right)U^{\gamma}}\right]\right\}\delta\left[\left(\vec{\mathbf{x}}-\vec{\mathbf{r}}\right)^2\right]$$
$$=2q\int d\tau\left\{-\frac{U_0U^{\beta}\left(x^{\alpha}-r^{\alpha}\right)\delta\left(x_0-r_0\right)}{\left(x_{\gamma}-r_{\gamma}\right)U^{\gamma}}+\frac{d}{d\tau}\left[\frac{\left(x^{\alpha}-r^{\alpha}\right)U^{\beta}}{\left(x_{\gamma}-r_{\gamma}\right)U^{\gamma}}\right]H\left(x_0-r_0\right)\right\}\delta\left[\left(\vec{\mathbf{x}}-\vec{\mathbf{r}}\right)^2\right] \tag{8.31}$$
$$=2q\int d\tau\left\{\frac{U_0U^{\beta}R^{\alpha}\delta\left(x_0-r_0\right)}{\left(x_{\gamma}-r_{\gamma}\right)U^{\gamma}}\delta\left(-R^2\right)+\frac{d}{d\tau}\left[\frac{\left(x^{\alpha}-r^{\alpha}\right)U^{\beta}}{\left(x_{\gamma}-r_{\gamma}\right)U^{\gamma}}\right]H\left(x_0-r_0\right)\delta\left[\left(\vec{\mathbf{x}}-\vec{\mathbf{r}}\right)^2\right]\right\},$$

but the first part of the integrand does not contribute to integral when $R\neq 0$, so

$$\partial^\alpha A^\beta = 2q\int d\tau \frac{d}{d\tau}\left\{\frac{\left[x^\alpha - r^\alpha(\tau)\right]U^\beta(\tau)}{\left[x_\gamma - r_\gamma(\tau)\right]U^\gamma(\tau)}\right\} H\left[x_0 - r_0(\tau)\right]\delta\left\{\left[\vec{\mathbf{x}} - \vec{\mathbf{r}}(\tau)\right]^2\right\}. \quad (8.32)$$

Upon using equation (8.21) to solve this integral, and equation (7.81) for the electromagnetic tensor, we find that

$$\boxed{F^{\alpha\beta} = \frac{q}{(x_\delta - r_\delta)U^\delta}\frac{d}{d\tau}\left\{\frac{\left[x^\alpha - r^\alpha(\tau)\right]U^\beta(\tau) - \left[x^\beta - r^\beta(\tau)\right]U^\alpha(\tau)}{\left[x_\gamma - r_\gamma(\tau)\right]U^\gamma(\tau)}\right\}\Bigg|_{\tau=\tau_0}} \quad (8.33)$$

With this result, we are now in a position to express the electric and magnetic induction field. Remembering that $R \equiv x_0 - r_0(\tau_0) = \left|\mathbf{x} - \mathbf{r}(\tau_0)\right|$, then

$$(x_\alpha - r_\alpha) = (R, -R\mathbf{n}), \quad (8.34)$$

and furthermore

$$U_\alpha = (\gamma c, -\gamma c\boldsymbol{\beta}). \quad (8.35)$$

Since

$$\frac{d\gamma}{d\tau} = \frac{d}{d\tau}(1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta})^{-1/2} = \gamma^3\boldsymbol{\beta}\cdot\frac{d\boldsymbol{\beta}}{d\tau} = \gamma^4\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}, \quad (8.36)$$

with $\dot{\boldsymbol{\beta}} = d\boldsymbol{\beta}/dt$, we have

$$\begin{aligned}
\frac{dU_\alpha}{d\tau} &= \left(c\gamma^4\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}, -c\gamma^4\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)\boldsymbol{\beta} - c\gamma^2\dot{\boldsymbol{\beta}}\right) \\
\frac{d}{d\tau}\left[\left(x_\gamma - r_\gamma\right)U^\gamma\right] &= -U_\gamma U^\gamma + \left(x_\gamma - r_\gamma\right)\frac{dU^\gamma}{d\tau} \\
&= -c^2 + \left(x_\gamma - r_\gamma\right)\frac{dU^\gamma}{d\tau}.
\end{aligned} \quad (8.37)$$

Using equations (8.37) and (8.23), it can be shown from equation (8.33) that

$$\boxed{\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= q\left[\frac{\mathbf{n} - \boldsymbol{\beta}}{\gamma^2(1 - \boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}\right]_{\text{ret}} + \frac{q}{c}\left[\frac{\mathbf{n}\times\left\{(\mathbf{n} - \boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}}{(1 - \boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]_{\text{ret}} \\
\mathbf{B}(\mathbf{x},t) &= \left[\mathbf{n}\times\mathbf{E}\right]_{\text{ret}}
\end{aligned}} \quad (8.38)$$

We see that the electromagnetic fields are composed of a velocity and an acceleration component, and only the latter radiates in the far field. Note also that in the radiation field $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{n}$ form a right-handed triad of mutually perpendicular vectors, and that $|\mathbf{E}| = |\mathbf{B}|$.

8.3 The Total Power Radiated by an Accelerated Charge

In order to simplify our derivation for the total power radiated by an accelerated charge, we need to make a short digression to discuss ways to evaluate the time component of any four-vector as measured by an observer with a given four-velocity.

8.3.1 Evaluating the time components of four-vectors

Since a four-velocity $\vec{\mathbf{U}}$ of a particle is expressed as

$$U^{\alpha} = \begin{pmatrix} \gamma c \\ \gamma \mathbf{u} \end{pmatrix} \tag{8.39}$$

when seen by a stationary observer, it takes a particularly simple form in the rest frame of the particle itself. That is, for an observer moving with the particle we have

$$U'^{\alpha} = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}, \tag{8.40}$$

where $\mathbf{0}$ is the null vector in three-space. Because of this, the time component of an arbitrary four-vector $\vec{\mathbf{A}}$ as measured in the rest frame of the particle is

$$A'^{0} = \frac{1}{c} U'_{\beta} A'^{\beta}, \tag{8.41}$$

but since $U'_{\beta} A'^{\beta} = \vec{\mathbf{U}} \cdot \vec{\mathbf{A}}$ is an invariant, we can write in general

$$\boxed{A'^{0} = \frac{1}{c} \vec{\mathbf{U}} \cdot \vec{\mathbf{A}}} \tag{8.42}$$

It is important to emphasize the fact that if A'^{0} is evaluated in the rest frame of the particle, the quantity $\vec{\mathbf{U}} \cdot \vec{\mathbf{A}}$ can be evaluated in any reference frame. For example, set $\vec{\mathbf{A}} = \vec{\mathbf{x}}$ in equation (8.42)

$$\begin{aligned} x'^0 &= \frac{1}{c}U^\alpha x_\alpha = \frac{1}{c}\frac{dx^\alpha}{d\tau}x_\alpha \\ &= \frac{1}{2c}\frac{d}{d\tau}\left(x^\alpha x_\alpha\right) = \frac{1}{2c}\frac{d\left(c^2\tau^2\right)}{d\tau} \\ &= c\tau, \end{aligned} \tag{8.43}$$

which is expected, since proper time is the time seen by an observer in the rest frame. Another example can be worked out if we set $\vec{\mathbf{A}} = \vec{\mathbf{k}}$, then

$$\begin{aligned} k'^0 &= \frac{1}{c}U^\alpha k_\alpha = \gamma\left(ck^0 - \mathbf{u}\cdot\mathbf{k}\right) \\ &= \gamma k^0\left[1 - \frac{uk}{ck^0}\cos(\theta)\right], \end{aligned} \tag{8.44}$$

and

$$\omega' = \gamma\omega\left[1 - \frac{u}{c}\cos(\theta)\right]. \tag{8.45}$$

Equation (8.45) is the Doppler shift formula.

8.3.2 Emission from Relativistic Particles

Let's consider a particle moving at relativistic velocities, and we define its *instantaneous rest frame* K' as the referential where the particle has zero velocity at a given time. We also define another frame K relative to which the particle is moving at velocity $-u$ (i.e., u is the instantaneous velocity of K', or the particle, as seen by K). We can use equation (8.42) to relate radiated power as seen in both frames. First, we consider the four-momentum $\vec{\mathbf{P}}$, and more precisely its time component P'^0 in the instantaneous rest frame of the particle

$$\begin{aligned} P'^0 &= \frac{1}{c}U^\alpha P_\alpha = \frac{\gamma}{c}\left(P^0 c - \mathbf{u}\cdot\mathbf{p}\right) \\ &= \frac{\gamma}{c}\left(P^0 c - \gamma m u^2\right) = \gamma P^0\left(1 - \frac{u^2}{c^2}\right) \\ &= \frac{P^0}{\gamma}, \end{aligned} \tag{8.46}$$

since $P^\alpha = mU^\alpha$. Similarly, we have

$$\begin{aligned} dx'^0 &= \frac{1}{c} U^\alpha dx_\alpha = \frac{\gamma}{c}\left(c dx^0 - \mathbf{u}\cdot d\mathbf{x}\right) \\ &= \gamma dx^0 \left(1 - \frac{u^2}{c^2}\right) = \frac{dx^0}{\gamma}. \end{aligned} \tag{8.47}$$

Equations (8.46) and (8.47) can obviously rewritten as

$$E_0 = \gamma E_0' \quad \text{and} \quad dt = \gamma dt'. \tag{8.48}$$

If we consider the amount of energy dW (or dW') radiated by the particle in an amount of time dt (or dt'), we can evaluate the power radiated P (or P') as

$$P = \frac{dW}{dt} \quad \text{and} \quad P' = \frac{dW'}{dt'}, \tag{8.49}$$

and from equations (8.48) we find that the total power emitted is a Lorentz invariant. That is,

$$P = P'. \tag{8.50}$$

We can use this invariance of the total power to our advantage. If we put ourselves in the rest frame K' of the particle, then we can consider the non-relativistic form for the equations of the electromagnetic fields. From equations (8.38), in the radiation field, we have

$$\mathbf{E}(\mathbf{x},t) = \frac{q}{c}\left[\frac{\mathbf{n}\times\left(\mathbf{n}\times\dot{\boldsymbol{\beta}}\right)}{R}\right]_{\text{ret}}, \tag{8.51}$$

which implies that the radiation is polarized in the plane containing the acceleration of the particle and $\mathbf{n}$. The instantaneous energy flux $\mathbf{S}$ is given by

$$\mathbf{S} = \frac{c}{4\pi}\left(\mathbf{E}\times\mathbf{B}\right) = \frac{c}{4\pi}\left|\mathbf{E}\right|^2 \mathbf{n}. \tag{8.52}$$

The power radiated per unit solid angle is

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{c}{4\pi}\left|R\mathbf{E}\right|^2 \\ &= \frac{q^2}{4\pi c}\left|\mathbf{n}\times\left(\mathbf{n}\times\dot{\boldsymbol{\beta}}\right)\right|^2. \end{aligned} \tag{8.53}$$

If we define θ as the angle between the acceleration $\dot{\mathbf{u}}$ and $\mathbf{n}$, then

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}|\dot{\mathbf{u}}|^2 \sin^2(\theta), \tag{8.54}$$

and the total instantaneous power radiated by an accelerated charge, in the non-relativistic limit, is given by the so-called **Larmor formula**

$$\boxed{P = \frac{2}{3}\frac{q^2}{c^3}|\dot{\mathbf{u}}|^2} \tag{8.55}$$

It is possible to generalize this result to the fully relativistic case if we remember that the four-acceleration $a^\mu \equiv dU^\mu / d\tau$ is "perpendicular" to the four-velocity (see equation (7.105)) with

$$a^\mu U_\mu = 0. \tag{8.56}$$

Then using equations (8.42) and (8.56) we find that

$$a'^0 = \frac{1}{c} a^\mu U_\mu = 0, \tag{8.57}$$

and with $\mathbf{a} = \dot{\mathbf{u}}$

$$\mathbf{a} \cdot \mathbf{a} = -a'^\mu a'_\mu = -a^\mu a_\mu = -\vec{\mathbf{a}} \cdot \vec{\mathbf{a}}. \tag{8.58}$$

Hence, we can write the generalized relativistic version of the Larmor formula in a manifestly covariant form as

$$\boxed{P = -\frac{2}{3}\frac{q^2}{c^3}\left(\frac{d\vec{\mathbf{U}}}{d\tau} \cdot \frac{d\vec{\mathbf{U}}}{d\tau}\right)} \tag{8.59}$$

Alternatively, we can write from the first of equations (8.37) that

$$\begin{aligned}
\frac{dU^\alpha}{d\tau}\frac{dU_\alpha}{d\tau} &= c^2\gamma^8\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2 - c^2\gamma^8\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2\beta^2 - c^2\gamma^4\left(\dot{\boldsymbol{\beta}}\right)^2 - 2c^2\gamma^6\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2 \\
&= c^2\gamma^6\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2 - c^2\gamma^4\left(\dot{\boldsymbol{\beta}}\right)^2 - 2c^2\gamma^6\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2 \\
&= -\left[c^2\gamma^4\left(\dot{\boldsymbol{\beta}}\right)^2 + c^2\gamma^6\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2\right] \\
&= -c^2\gamma^6\left\{\left(\dot{\boldsymbol{\beta}}\right)^2 - \left[\left(\boldsymbol{\beta}\right)^2\left(\dot{\boldsymbol{\beta}}\right)^2 - \left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2\right]\right\} \\
&= -c^2\gamma^6\left[\left(\dot{\boldsymbol{\beta}}\right)^2 - \left(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right)^2\right],
\end{aligned} \tag{8.60}$$

and finally,

$$\boxed{P=\frac{2}{3}\frac{q^2}{c}\gamma^6\left[\left(\dot{\boldsymbol{\beta}}\right)^2-\left(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right)^2\right]} \tag{8.61}$$

We see that the power radiated is composed of parts proportional and perpendicular to the acceleration of the particle.

8.4 Angular Distribution of Radiation Emitted by an Accelerated Charge

In the non-relativistic approximation, i.e., equation (8.54), the angular distribution of radiation is $\sin^2(\theta)$, where θ is the angle between the acceleration $\dot{\mathbf{u}}$ and the unit vector $\mathbf{n}$. In the fully relativistic case, we must use the radiation components of equations (8.38) for the electromagnetic fields to calculate the energy flux emitted in the direction of $\mathbf{n}$. That is,

$$\begin{aligned}\left[\mathbf{S}\cdot\mathbf{n}\right]_{\text{ret}} &= \frac{c}{4\pi}\left[\mathbf{n}\cdot\left(\mathbf{E}\times\mathbf{B}\right)\right]_{\text{ret}}\\ &= \frac{c}{4\pi}\left[\mathbf{n}\cdot\left(\mathbf{E}\times\left[\mathbf{n}\times\mathbf{E}\right]\right)\right]_{\text{ret}}\\ &= \frac{c}{4\pi}\left[\mathbf{n}\cdot\left(\mathbf{n}\left|\mathbf{E}\right|^2-\mathbf{E}\left[\mathbf{n}\cdot\mathbf{E}\right]\right)\right]_{\text{ret}}\\ &= \frac{c}{4\pi}\left[\left|\mathbf{E}\right|^2\right]_{\text{ret}},\end{aligned} \tag{8.62}$$

since $\mathbf{n}\cdot\mathbf{E}=0$ for the radiation field (see equations (8.38)). Finally, the flux is

$$\left[\mathbf{S}\cdot\mathbf{n}\right]_{\text{ret}} = \frac{q^2}{4\pi c}\left[\frac{1}{R^2}\left|\frac{\mathbf{n}\times\left\{\left(\mathbf{n}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right\}}{\left(1-\boldsymbol{\beta}\cdot\mathbf{n}\right)^3}\right|^2\right]_{\text{ret}}. \tag{8.63}$$

Although the energy flux expressed in equation (8.63) is evaluated at the retarded time $t'=t-R(t')/c$, it reaches the observer located at $\mathbf{x}$ at time t. The energy per unit area measured by the observer in a period of time going from $T_1+R(T_1)/c$ and $T_2+R(T_2)/c$ is given by

$$\int_{t=T_1+R(T_1)/c}^{t=T_2+R(T_2)/c}\left[\mathbf{S}\cdot\mathbf{n}\right]_{\text{ret}}\,dt = \int_{t'=T_1}^{t'=T_2}\left(\mathbf{S}\cdot\mathbf{n}\right)\frac{dt}{dt'}\,dt', \tag{8.64}$$

where t' is the retarded time as measured in the inertial frame of the observer, and *not* the time in the frame where the charge is at rest (as it appears in Lorentz transformations, for example). Since $dt = (1 - \boldsymbol{\beta} \cdot \mathbf{n}) dt'$ (see equation (4.63) of the lecture notes, and equation (6.19) of the second list of problem), we can write the power radiated per unit angle as

$$\frac{dP(t')}{d\Omega} = R^2 (\mathbf{S} \cdot \mathbf{n})(1 - \boldsymbol{\beta} \cdot \mathbf{n}). \tag{8.65}$$

If we further assume that the amplitude of the displacements of the charge are small compared to R, then both $\mathbf{n}$ and R are (approximately) unaffected by its motion and

$$\boxed{\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left|\mathbf{n} \times \left\{(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right\}\right|^2}{(1 - \boldsymbol{\beta} \cdot \mathbf{n})^5}} \tag{8.66}$$

8.4.1 **Acceleration Parallel to the Velocity**

When a charge exhibits a linear motion and its acceleration is parallel to its velocity, then equation (8.66) becomes

$$\frac{dP_{\|}(t')}{d\Omega} = \frac{q^2 |\dot{\mathbf{u}}|^2}{4\pi c^3} \frac{\sin^2(\theta)}{\left[1 - \beta \cos(\theta)\right]^5}, \tag{8.67}$$

where θ is, once again, the angle between $\mathbf{n}$ and $\boldsymbol{\beta}$ (or $\dot{\boldsymbol{\beta}}$). Equation (8.67) simplifies to equation (8.54) in the non-relativistic limit, but as $\beta \to 1$ the angular distribution is tipped forward and increases in amplitude, as show in Figure 8.1. Setting the θ-derivative of equation (8.67) to zero, we find that the angle of maximum intensity $\theta_{\max}$ is given by

$$\theta_{\max} = \cos^{-1}\left[\frac{1}{3\beta}\left(\sqrt{1 + 15\beta^2} - 1\right)\right]. \tag{8.68}$$

In the extremely relativistic limit this relation becomes

$$\lim_{\beta \to 1} \theta_{\max} = \frac{1}{2\gamma}, \tag{8.69}$$

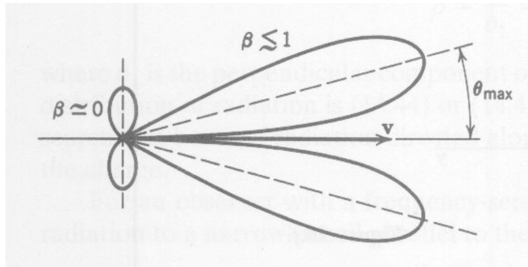

Figure 8.1 – Radiation pattern for a charge accelerated along its direction of motion. The two patterns are not to scale, the relativistic one ($\gamma \sim 2$) has been reduced by a factor of about 100 for the same acceleration.

since $\beta = \left(1 - 1/\gamma^2\right)^{1/2} \simeq 1 - 1/\left(2\gamma^2\right)$. Because the angular distribution of the radiation is highly peaked, we can write $\cos(\theta) \simeq 1 - \theta^2/2$, and from equation the previous approximation for β, we find that

$$1 - \beta\cos(\theta) \simeq 1 - \left(1 - \frac{1}{2\gamma^2}\right)\left(1 - \frac{\theta^2}{2}\right) \simeq \frac{1 + \gamma^2\theta^2}{2\gamma^2}, \tag{8.70}$$

and thus

$$\frac{dP_{\|}(t')}{d\Omega} \simeq \frac{8}{\pi}\frac{q^2\left|\dot{\mathbf{u}}\right|^2}{c^3}\gamma^8 \frac{\gamma^2\theta^2}{\left[1 + \gamma^2\theta^2\right]^5}. \tag{8.71}$$

8.4.2 Acceleration Perpendicular to the Velocity

If the orientation of $\mathbf{n}, \boldsymbol{\beta}$, and $\dot{\boldsymbol{\beta}}$ as a function of the angles θ and φ is as defined in Figure 8.2, then equation (8.66) becomes

$$\begin{aligned}
\frac{dP_{\perp}(t')}{d\Omega} &= \frac{q^2}{4\pi c}\frac{\left|\mathbf{n}\times\left(\mathbf{n}\times\dot{\boldsymbol{\beta}}\right) - \mathbf{n}\times\left(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right)\right|^2}{\left(1 - \boldsymbol{\beta}\cdot\mathbf{n}\right)^5} \\
&= \frac{q^2}{4\pi c}\frac{\left|\mathbf{n}\left(\mathbf{n}\cdot\dot{\boldsymbol{\beta}}\right) - \dot{\boldsymbol{\beta}} - \boldsymbol{\beta}\left(\mathbf{n}\cdot\dot{\boldsymbol{\beta}}\right) + \dot{\boldsymbol{\beta}}\left(\mathbf{n}\cdot\boldsymbol{\beta}\right)\right|^2}{\left(1 - \boldsymbol{\beta}\cdot\mathbf{n}\right)^5} \\
&= \frac{q^2\left|\dot{\mathbf{u}}\right|^2}{4\pi c^3}\frac{\left|\sin(\theta)\cos(\varphi)\left[\mathbf{n} - \beta\mathbf{e}_z\right] - \left[1 - \beta\cos(\theta)\right]\mathbf{e}_x\right|^2}{\left[1 - \beta\cos(\theta)\right]^5},
\end{aligned} \tag{8.72}$$

and after calculating the required scalar products

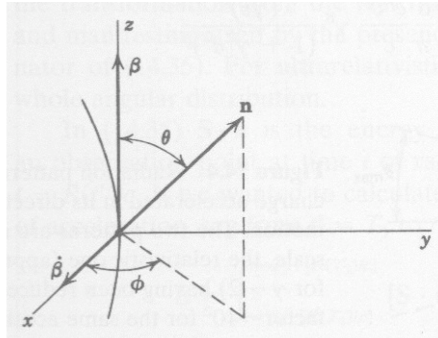

Figure 8.2 – The orientation of $\mathbf{n}, \boldsymbol{\beta}$, and $\dot{\boldsymbol{\beta}}$ as a function of the angles θ and φ for the case of a charge accelerated in a direction perpendicular to its velocity.

$$\frac{dP_{\perp}(t')}{d\Omega} = \frac{q^2}{4\pi c^3} \frac{|\dot{\mathbf{u}}|^2}{\left[1-\beta\cos(\theta)\right]^3} \left\{1 - \frac{\sin^2(\theta)\cos^2(\varphi)}{\gamma^2\left[1-\beta\cos(\theta)\right]^2}\right\}. \tag{8.73}$$

Again, if we concentrate on the extreme relativistic limit where $\beta \to 1$, then inserting equation (8.70) into equation (8.72) we get

$$\frac{dP_{\perp}(t')}{d\Omega} \simeq \frac{2}{\pi} \frac{q^2}{c^3} \gamma^6 \frac{|\dot{\mathbf{u}}|^2}{\left[1+\gamma^2\theta^2\right]^3} \left[1 - \frac{4\gamma^2\theta^2\cos^2(\varphi)}{\left(1+\gamma^2\theta^2\right)^2}\right]. \tag{8.74}$$

The angular distribution of the radiation is mostly contained within a range of angles approximately given by $\theta \sim 1/\gamma$, as shown in Figure 8.3.

8.4.3 *Distribution in Frequency and Angle of Energy Radiated by Accelerating Charges*

Looking back at equation (8.62), we could equation write the following for the expression of the power radiated per unit solid angle as function of *observer's time* (not the retarded as was done in the last section)

$$\frac{dP(t)}{d\Omega} = |\mathbf{A}(t)|^2, \tag{8.75}$$

with

$$\mathbf{A}(t) = \left(\frac{c}{4\pi}\right)^{1/2} \left[R\mathbf{E}\right]_{\text{ret}}. \tag{8.76}$$

Integrating equation (8.75) over time yields the energy radiated per solid angle

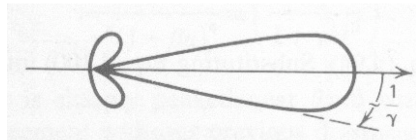

Figure 8.3 – The angular distribution of radiation emitted by a particle with perpendicular acceleration and velocity (the acceleration is directed "upward").

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} \left|\mathbf{A}(t)\right|^2 \, dt. \tag{8.77}$$

If we now introduce the Fourier transform pair

$$\begin{aligned} \mathbf{A}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{A}(t) e^{i\omega t} \, dt \\ \mathbf{A}(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{A}(\omega) e^{-i\omega t} \, d\omega, \end{aligned} \tag{8.78}$$

(please note that if $\mathbf{A}(t)$ is real, then $\mathbf{A}(\omega) = \mathbf{A}^*(-\omega)$), then equation (8.77) can be transformed to

$$\begin{aligned} \frac{dW}{d\Omega} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \, \mathbf{A}(\omega') \mathbf{A}^*(\omega) \int_{-\infty}^{\infty} dt \, e^{i(\omega'-\omega)t} \\ &= \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \, \mathbf{A}(\omega') \mathbf{A}^*(\omega) \delta(\omega' - \omega) \\ &= \int_{-\infty}^{\infty} \left|\mathbf{A}(\omega)\right|^2 \, d\omega. \end{aligned} \tag{8.79}$$

The general result that

$$\int_{-\infty}^{\infty} \left|\mathbf{A}(\omega)\right|^2 \, d\omega = \int_{-\infty}^{\infty} \left|\mathbf{A}(t)\right|^2 \, dt \tag{8.80}$$

is called **Parseval's theorem**. We now define a new quantity for the radiated energy per solid unit angle per unit frequency as

$$\frac{d^2 I(\omega, \mathbf{n})}{d\Omega \, d\omega} = \left|\mathbf{A}(\omega)\right|^2 + \left|\mathbf{A}(-\omega)\right|^2, \tag{8.81}$$

which for a real signal becomes

$$\frac{d^2 I(\omega, \mathbf{n})}{d\Omega \, d\omega} = 2\left|\mathbf{A}(\omega)\right|^2. \tag{8.82}$$

Inserting equation (8.81) (or (8.82)) in equation (8.79), we find

$$\frac{dW}{d\Omega} = \int_0^\infty \frac{d^2 I(\omega,\mathbf{n})}{d\Omega\, d\omega}\, d\omega. \tag{8.83}$$

To apply equation (8.83) to the problem of an accelerated charge, we first calculate the Fourier transform of the radiation electric field in the far field (from the first of equations (8.38))

$$\mathbf{A}(\omega) = \left(\frac{q^2}{8\pi^2 c}\right)^{1/2} \int_{-\infty}^{\infty} e^{i\omega t} \left[\frac{\mathbf{n}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\text{ret}} dt, \tag{8.84}$$

where the quantity in between the brackets is to be evaluated at the retarded time $t' = t - R(t')/c$. We now change the variable of integration from t to t' just as we did for equation (8.64), and we get

$$\mathbf{A}(\omega) = \left(\frac{q^2}{8\pi^2 c}\right)^{1/2} \int_{-\infty}^{\infty} e^{i\omega[t' + R(t')/c]} \frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2}\, dt', \tag{8.85}$$

and we, again, assume that the amplitude of the displacements of the charge are small compared to R so that both $\mathbf{n}$ and R are (approximately) unaffected by its motion. Furthermore, if we define the distance between the observation point and the origin from which the charge's position $\mathbf{r}(t')$ is evaluated as x, then we can write

$$R(t') \simeq x - \mathbf{n}\cdot\mathbf{r}(t'), \tag{8.86}$$

and

$$\mathbf{A}(\omega) = \left(\frac{q^2}{8\pi^2 c}\right)^{1/2} e^{i\omega x/c} \int_{-\infty}^{\infty} e^{i\omega[t' - \mathbf{n}\cdot\mathbf{r}(t')/c]} \frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2}\, dt'. \tag{8.87}$$

We should note that the exponential term in front of the integral has no physical significance and can be neglected, since the quantity we seek to evaluate is proportional to $|\mathbf{A}(\omega)|^2$. Inserting equation (8.87) into equation (8.82), we get

$$\boxed{\frac{d^2 I(\omega,\mathbf{n})}{d\omega\, d\Omega} = \frac{q^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} e^{i\omega[t' - \mathbf{n}\cdot\mathbf{r}(t')/c]} \frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2}\, dt' \right|^2} \tag{8.88}$$

Given the equation for the motion of the charge $\mathbf{r}(t')$, this result (i.e., equation (8.88)) can be used to calculate the radiated energy per unit solid angle per unit frequency. Alternatively, we can modify equation (8.87) by integrating by parts, since the ratio involving $\mathbf{n}, \boldsymbol{\beta}$, and $\dot{\boldsymbol{\beta}}$ is a perfect time differential. More precisely,

$$\begin{aligned}
\frac{d}{dt'}\left[\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right] &= \frac{\left[\mathbf{n}\times\left(\mathbf{n}\times\dot{\boldsymbol{\beta}}\right)\right](1-\boldsymbol{\beta}\cdot\mathbf{n})+\left(\dot{\boldsymbol{\beta}}\cdot\mathbf{n}\right)\left[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} \\
&= \frac{\mathbf{n}\times\left(\mathbf{n}\times\dot{\boldsymbol{\beta}}\right)-\left[\left(\dot{\boldsymbol{\beta}}\cdot\mathbf{n}\right)\mathbf{n}-\dot{\boldsymbol{\beta}}\right](\boldsymbol{\beta}\cdot\mathbf{n})+\left[(\boldsymbol{\beta}\cdot\mathbf{n})\mathbf{n}-\boldsymbol{\beta}\right]\left(\dot{\boldsymbol{\beta}}\cdot\mathbf{n}\right)}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} \\
&= \frac{\mathbf{n}\times\left(\mathbf{n}\times\dot{\boldsymbol{\beta}}\right)+\dot{\boldsymbol{\beta}}(\boldsymbol{\beta}\cdot\mathbf{n})-\boldsymbol{\beta}\left(\dot{\boldsymbol{\beta}}\cdot\mathbf{n}\right)}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} \\
&= \frac{\mathbf{n}\times\left(\mathbf{n}\times\dot{\boldsymbol{\beta}}\right)-\mathbf{n}\times\left(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right)}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} \\
&= \frac{\mathbf{n}\times\left\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2},
\end{aligned} \tag{8.89}$$

so that we can write

$$\begin{aligned}
\mathbf{A}(\omega) &= \left(\frac{q^2}{8\pi^2 c}\right)^{1/2}\int_{-\infty}^{\infty} e^{i\omega\left[t'-\mathbf{n}\cdot\mathbf{r}(t')/c\right]}\frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2}\,dt' \\
&= -i\omega\left(\frac{q^2}{8\pi^2 c}\right)^{1/2}\int_{-\infty}^{\infty} e^{i\omega\left[t'-\mathbf{n}\cdot\mathbf{r}(t')/c\right]}(1-\boldsymbol{\beta}\cdot\mathbf{n})\left[\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right]dt' \\
&= -i\omega\left(\frac{q^2}{8\pi^2 c}\right)^{1/2}\int_{-\infty}^{\infty} e^{i\omega\left[t'-\mathbf{n}\cdot\mathbf{r}(t')/c\right]}\left[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right]dt',
\end{aligned} \tag{8.90}$$

where it was assumed that $\boldsymbol{\beta}$ vanishes at $t' = \pm\infty$. Inserting equation (8.90) into equation (8.82) we finally find that

$$\boxed{\frac{d^2 I(\omega,\mathbf{n})}{d\omega\, d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} e^{i\omega\left[t'-\mathbf{n}\cdot\mathbf{r}(t')/c\right]}\left[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\right]dt'\right|^2} \tag{8.91}$$